Test 2 Numerical Mathematrics 2
December, 2018

Duration: 1 hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark for this test.

1. [1.5] Without actually computing the eigenvalues, localize the eigenvalues of the matrix below using the Gershgorin theorems and general properties of the matrix,

$$\begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Solution: Observe that the matrix is reducible, hence the eigenvalues of the matrix are the union of the eigenvalues of the first 3x3 diagonal block and the last 2x2 diagonal block [0.5].

The 2x2 block is symmetric, hence its eigenvalues are real. The circles of the first block all coincide. So also every point on the boundary can be an eigenvalue. So the eigenvalues are on and inside the circle with center 2 and radius 2 [0.5].
In fact 0 and 4 are eigenvalues with eigenvectors [1;1;1] and [1;-1;1], respectively.

The last 2x2 block has two circles with radia 1 and 2 and 1 as centerpoints. There is no point in common on the real axis so we have that the corresponding eigenvalues will be in (0,3). [0.5] So concluding there will be 2 eigenvalues on (0,3) and 3 in the circle with center 2 and radius 2.

2. (a) [1.5] Let μ be an eigenvalue of $A+E$, moreover, A has a complete set of eigenvectors. Show that

$$\min_{\lambda \in \sigma(A)} |\lambda - \mu| \leq C||E||$$

where the matrix norms used are induced by a vector norm and $C > 0$. Also give C

Solution: We have $(A+E)x = \mu x$. Since, A has a complete set of eigenvector, it is diagonizable. Let $P^{-1}AP = D$, then we find by premultiplication of P^{-1}

$$(D + P^{-1}EP)P^{-1}x = \mu P^{-1}x$$

which is equivalent to

$$(D - \mu I)(P^{-1}x) = -P^{-1}EP(P^{-1}x). \qquad [0.5]$$

Now we assume that μ is not equal to an eigenvalue of A, and hence $(D - \mu I)$ will be nonsingular. So we may write

$$P^{-1}x = -(D - \mu I)^{-1}P^{-1}EP(P^{-1}x)$$

Take norms and use associated inequalities to find

$$||P^{-1}x|| \leq ||(D - \mu I)^{-1}||\ \kappa(P)\ ||E||\ ||(P^{-1}x)||$$

which is equivalent to

$$1 \leq ||(D - \mu I)^{-1}||\ \kappa(P)\ ||E||. \qquad [0.5]$$

Now it holds that the norm of a diagonal matrix is just the maximum element in abs. sense on the diagonal. So $||(D - \mu I)^{-1}|| = \max_{\lambda \in \sigma(A)} 1/|\lambda - \mu| = 1/\min_{\lambda \in \sigma(A)} |\lambda - \mu|$. After insertion into the last expression we just have to multiply by $\min_{\lambda \in \sigma(A)} |\lambda - \mu|$ to get the desired expression. So $C = \kappa(P)$ [0.4]. Observe that for the case that μ is equal to an eigenvalue of A the inequality is satisfied too [0.1].

(b) [0.5] What does the expression in the previous part mean?

Solution: It means that we can bound the perturbation in an eigenvalue by the perturbation in A. Moreover, C is the condition number in this case.

3. (a) [1.0] Suppose the matrix A of order 100 has eigenvalues $\lambda_i = i^2$, for $i = 1, \cdots 100$. If we apply shift 1000 in inverse iteration, to which eigenvector of A will the method converge to?

Solution: The eigenvalues of the shifted and inverted matrix are $1/(i^2 - 1000)$. The eigenvalue of A closest to 1000 is $32^2 = 1024$ and the next closest is $31^2 = 961$. The power method will converge to the eigenvector associated to the biggest eigenvalue of the shifted and and inverted matrix, which means for $\lambda_{32} = 1024$ of the original matrix.

(b) [0.5] What is the speed of convergence?

Solution: The speed of convergence is the ratio of the one but largest and the largest eigenvalue of the shifted and inverted matrix, which is in this case 24/39=8/13.

4. [1.0] Give the Householder matrix that transforms the matrix

$$\begin{bmatrix} 2 & -3 & -4 \\ -3 & 2 & -1 \\ -4 & -1 & 2 \end{bmatrix}$$

into tridiagonal form and show how it is used. So you don't need to make the matrix tridiagonal, but merely convince us how this should be done.

Solution: In this case, we only have to deal with the last two entries of the first column, i.e. $u = -[3;4]$. We should map this on the x-axis by a mirroring. We choose to map it to $[5;0]$. From this the unnormalized v in $H = I - 2vv^T$ is $\hat{v} = [8;4] = 4[2;1]$ which has length $4\sqrt{5}$. So we have $v = \frac{1}{\sqrt{5}}[2;1]$. Now we have to pre- and postmultiply by diag({1,H})

$$\begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 2 & u^T \\ u & A_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} 2 & w^T \\ w & HA_{22}H \end{bmatrix}.$$

This results in a matrix of shape

$$\begin{bmatrix} 2 & 5 & 0 \\ 5 & * & * \\ 0 & * & * \end{bmatrix}$$

where the lower 2x2 block is

$$H \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} H.$$

5. Consider the three matrices below

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4.6792 & .2979 & 0 \\ .2979 & 3.0524 & .0274 \\ 0 & .0274 & 1.2684 \end{bmatrix}, \quad \begin{bmatrix} 4.7104 & .1924 & 0 \\ .1924 & 3.0216 & -.0115 \\ 0 & -.0115 & 1.2680 \end{bmatrix}$$

which are respectively the original matrix and two subsequent iterates in the QR-method. Moreover, it is given that the eigenvalues of the original matrix are 4.7321, 3.0 and 1.2679.

(a) [0.7] How is the QR-method defined? Where does it, for general real matrices, converge to?

Solution: Set $A^{(0)} = A$ and perform for $i = 0, 1, 2, 3, \cdots$ the iteration: (i) compute a QR-factorization of $A^{(i)} = A$, giving $Q^{(i)}$ and $R^{(i)}$, (ii) Compute $A^{(i+1)} = R^{(i)}Q^{(i)}$. For general matrices this converges to the real Schur form of the matrix A, which may have some 2x2 blocks on the diagonal if the matrix has complex eigenvalues.

(b) [0.6] Explain the reduction factor of the off-diagonal elements from the middle to the right matrix.

Solution: The (2,1) coefficient should decrease by about $\lambda_2/\lambda_1 \approx 3/4.7 \approx 0.6$ as it does, and the (3,2) coefficient by $\lambda_3/\lambda_2 \approx 1.3/3 = 0.43$ which it does too.

(c) [0.7] Suppose we apply a QR-step including shift to the middle matrix. By which factor will the (3,2) element decrease approximately?

Solution: We will shift with the last element of the matrix and get the new relevant eigenvalues: 4.73-1.27=3.46, 3.00-1.27=1.73, 1.2679-1.2684=-0.0005. So the element (3,2) will decrease by a factor $5.e-4/1.73 \approx 3.e-4$.

6. [1.0] Show that the orthogonalization of the basis of the Krylov subspace for a real and symmetric matrix A leads to its tridiagonal Galerkin approximation on that space.

Solution: For the orthogonalization we use the Gramm-Schmidt process during the construction of the space. This amounts to the expression

$$A[v_1\ v_2\ \cdots\ v_m] = [v_1\ v_2\ \cdots\ v_m\ v_{m+1}]H_{m+1,m}$$

where v_1 is the given vector for the Krylov space. Moreover v_{m+1} is constructed from Av_m which is orthogonalized with respect to v_i, $i = 1, \cdots, m$. Which leads to a matrix $H_{m+1,m}$ of Hessenberg form. If we now premultiply with the transpose of $[v_1\ v_2\ \cdots\ v_m]$ we obtain

$$[v_1\ v_2\ \cdots\ v_m]^T A[v_1\ v_2\ \cdots\ v_m] = [I, 0]H_{m+1,m} = H_{m,m}$$

Since the left-hand side is symmetric also $H_{m,m}$ must be symmetric. Since it also is of Hessenberg form it must be tridiagonal.